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Geometric phase decomposition for exactly solvable cases with interaction

R Aleksiejunas† and V Ivaska

Department of Radiophysics, Vilnius University, Sauletekio al. 9 k. 3, 2040 Vilnius, Lithuania

E-mail: `rimvydas.aleksiejunas@ff.vu.lt`

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Abstract. In the paper a problem of geometric phase decomposition for general evolutions in the Hilbert space is addressed. The decomposition of total phase into dynamical and geometrical parts employs a state representation in the noninteraction picture. The noninteraction picture is introduced with the help of decomposition of the system evolution operator into two parts, the one pertaining to free evolution and the other to interaction. The procedure requires the problem to be an exactly solvable case of the Schrödinger equation. The most common class of such problems includes dipole Hamiltonians, for which the evolution operator can be decomposed into a combination of unitary operators. Geometric phase decomposition in the noninteraction picture can be applied to general noncyclic evolutions, but for the cyclic states it reduces to the Floquet decomposition. Defined this way, geometric phase possesses characteristic features of geometric phase for the free system, but extends some stationary properties to temporal dependences. As a case, the time-dependent relationship between geometric phase and the nonstationarity of a quantum state is illustrated by an example of a spin-$\frac{1}{2}$ in a rotating magnetic field. It is shown that geometric phase reaches a maximum when the spin state becomes completely nonstationary.

1. Introduction

Geometric phase was introduced by Berry [1] as a phase factor arising after a system completes a closed path in the parametric space $\boldsymbol{R}$. Although geometric phase is independent of path parametrization $\{\boldsymbol{R}(t), 0 < t < T\}$, some misconceptions arise in choosing a particular parameter space. The problem is caused by the fact that geometric phase is associated with the evolution of a quantum system and not with a particular Hamiltonian used to achieve the evolution. Aharonov and Anandan [2] uniquely defined geometric phase for a given *projection* of the evolution on the projective Hilbert space $\mathcal{P}$. A given cyclic evolution on the projective space $\mathcal{P}$ corresponds to the infinite number of possible motions along the curves in the Hilbert space $\mathcal{H}$ and the possible Hamiltonians which propagate the state along these curves. Hence a procedure of obtaining geometric phase for a given quantum evolution seems to be nonunique. Every geometric phase decomposition in the Hilbert space employs a particular condition, such as parallel transport or cyclicity of the initial state, in order to define geometric phase uniquely.

In this paper an unambiguous geometric phase decomposition is constructed with the help of a new condition. This enables determination of the geometric phase temporal dependences for incomplete cyclic, as well as for general noncyclic evolutions, which is not always possible

† Corresponding author.

for other known decompositions. The system evolution operator is split into two parts, the one corresponding to free motion of a system and the other owing to interaction. Geometric phase is defined in the basis which initially coincides with the eigenstates of the free Hamiltonian, and the time evolution of the basis is governed by interaction. The procedure employs state vector representation which can be called the noninteraction picture, since it transforms away interaction evolution from the total evolution operator. The separation of the total evolution operator into parts requires the problem to be an exactly solvable case of the Schrödinger equation. The noninteraction picture has already been introduced in [3], considering geometric phase for a scattering wavefunction. In a similar manner we propose a decomposition which is valid for all noncyclic evolutions in the Hilbert space, while for the cyclic states it transforms into the Floquet decomposition, as a special case. Defined this way, geometric phase possesses characteristic features of geometric phase for the free system, but extends some stationary properties to temporal dependences. As a case, the time-dependent relationship between geometric phase and the nonstationarity of a quantum state is illustrated by the example of a spin-$\frac{1}{2}$ in a rotating magnetic field.

2. Nonuniqueness of geometric phase decomposition for evolutions in the Hilbert space

In this section we give a brief account of the problem of geometric phase nonuniqueness based on the literature. Geometric phase is defined uniquely only for a given projection of evolution on the projective Hilbert space $\mathcal{P}$ [2]. But there still exists freedom to choose a particular Hamiltonian which propagates a system along the curve in the Hilbert space $\mathcal{H}$. Consequently determination of geometric phase for a given Hamiltonian or evolution in the Hilbert space is nonunique. As was pointed out in [4, 5], all definitions of geometric phase for evolutions in the Hilbert space employ particular conditions, imposed on the system evolution operator or on its part $U'(t)$, which govern the evolution of the instantaneous eigenstate

$$|m(t)\rangle = U'(t)|m(0)\rangle \tag{1}$$

of the Hamiltonian $H(t)$. The most widely employed is the parallel transport condition [6–8]

$$\langle m(0)|U'^{+}(t)\,\dot{U}'(t)|m(0)\rangle = 0. \tag{2}$$

The other is a cyclicity condition for the initial state:

$$|m(T)\rangle = |m(0)\rangle \tag{3}$$

which was originally introduced by Berry [1]. The condition (3) for eigenstates is equivalent to the periodicity requirement for the evolution operator

$$U'(T) = U'(0) = I \tag{4}$$

which plays a key role in the Floquet decomposition of geometric phase [9–13] (in the usual notation: $Z(T) = Z(0) = I$). However, even condition (4) does not ensure the uniqueness of geometric phase, since Floquet decomposition is not unique by itself [13].

The difference between these two approaches to define geometric phase for the trajectories in the Hilbert space $\mathcal{H}$ can be obtained from the effect that conditions (2) and (3) cause on the geometric phase decomposition for a particular trajectory in projective space $\mathcal{P}$. For a given path parametrization $C : \{\boldsymbol{R}(t'), 0 < t' < t\}$ in $\mathcal{P}$, geometric phase for general evolutions is expressed as [14]

$$\gamma_m[C] = \arg\langle m; \boldsymbol{R}(0)|m; \boldsymbol{R}(t)\rangle + \mathrm{i}\int_0^t \mathrm{d}t'\, \dot{\boldsymbol{R}}(t')\cdot\langle m; \boldsymbol{R}(t')|\nabla_{\boldsymbol{R}}|m; \boldsymbol{R}(t')\rangle \tag{5}$$

where $|m; \boldsymbol{R}(t)\rangle$ parametrizes eigenvector $|m(t)\rangle$ of the Hamiltonian. It is obvious that condition (2) eliminates the second term in the geometric phase expression (5), while (3) for time moment $t = T$ eliminates the first. This means that in the first case geometric phase is defined as a phase of complex number $\langle m; \boldsymbol{R}(0)|m; \boldsymbol{R}(t)\rangle$, depending only on the initial and final points of the curve C in $\mathcal{P}$. In the second case, geometric phase is defined as an integral along the curve C in the parametric space $\mathcal{P}$.

The above cases clearly show the nonuniqueness of the geometric phase definition for a particular Hamiltonian or evolution operator. There exists yet another issue concerning ambiguity in defining geometric phase. This involves unitary transformations of the system state, which also act on the geometric phase. Geometric phase decomposition is not invariant under these transformations in the sense that the original geometric phase can be moved into transformed dynamical phase. By the proper choice of transformation, geometric phase can be completely removed, while dynamical phase retains the properties of geometric phase [5, 15]. It follows that dynamical phase also changes under unitary transformations. Indeed, with the help of particular transformations, dynamical phase could be modified in a way suitable for convenient separation of geometric phase [16].

Ambiguity in geometric phase decomposition due to unitary transformations of the system state could be removed by specifying the frame with respect to which the state vector is defined (for details, see the discussion in [5]). This aspect is sometimes overlooked, especially when comparing results of theoretical and experimental investigations of geometric phase. The situation is especially evident in determining geometric phases for spin systems in a rotating magnetic field. While the majority of theoretical investigations have been carried out in the static laboratory frame [17, 18], some experiments verify the existence of geometric phase in the rotating frame [19, 20]. As it was pointed out in [5], however, such decompositions are not identical.

3. Geometric phase decomposition in the noninteraction picture

First we start with the free Hamiltonian and determine geometric phase in the ordinary way as a difference between total and dynamical phases. Then we will introduce interaction and define geometric phase in the noninteraction picture.

Consider a system that is governed by a time-independent free Hamiltonian H_0 and corresponding evolution operator $U_0(t)$:

$$U_0(t) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} H_0 t}. \tag{6}$$

If the system is initially prepared in the state $|\Psi_0(0)\rangle$, then evolution of the state vector in time is given by

$$|\Psi_0(t)\rangle = U_0(t)|\Psi_0(0)\rangle. \tag{7}$$

Geometric phase for general evolutions, according to [21], is expressed as a difference between the total and dynamical phases:

$$\gamma_0(t) = \arg\langle\Psi_0(0)|\Psi_0(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \langle\Psi_0(t')|H_0|\Psi_0(t')\rangle \tag{8a}$$

$$= \arg\langle\Psi_0(0)|U_0(t)|\Psi_0(0)\rangle + \frac{t}{\hbar}\langle\Psi_0(0)|H_0|\Psi_0(0)\rangle. \tag{8b}$$

This is an expression of geometric phase for a free system. Suppose that the system is initially prepared in the eigenstate of H_0:

$$|\Psi_0(0)\rangle = |m\rangle \tag{9}$$

$$H_0|m\rangle = E_m|m\rangle \tag{10}$$

and at the time moment $t = 0$ interaction is turned on. Now evolution of the system is governed by the full Hamiltonian $H(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \frac{\mathrm{i}}{\hbar}H(t)|\Psi(t)\rangle \tag{11}$$

and corresponding evolution operator $U(t)$:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle \qquad U(0) = I \tag{12}$$

where $|\Psi(0)\rangle = |\Psi_0(0)\rangle = |m\rangle$.

To proceed further we construct a reference basis with respect to which geometric phase will be defined. We will require that geometric phase decomposition for a system with interaction reduces to geometric phase for the free system in the case where the interaction vanishes out. To fulfil such a condition, the reference basis must coincide with the eigenbasis of the free Hamiltonian for the initial time moment and evolve with time under the influence of interaction. For this purpose we need an interaction evolution operator in an explicit form. Since interaction generally is described by a time-dependent Hamiltonian, we cannot obtain the interaction evolution operator simply by exponentiating the interaction's Hamiltonian as in the case of free evolution (6). Therefore we restrict our analysis to a class of evolutions for which the interaction contribution to the evolution operator can be explicitly separated in the two possible ways:

$$U(t) = U_1(t)U_0(t) \tag{13}$$

or

$$U(t) = U_0(t)U_1(t) \tag{14}$$

where $U_1(t)$ is evolution operator due to interaction. These two decompositions of evolution operator represent exactly solvable cases of the Schrödinger equation (11). Note that, formally, it is possible to build operatorial decompositions (13) and (14) with the help of substitutions $U_1(t) = U(t)U_0^+(t)$ and $U_1(t) = U_0^+(t)U(t)$, respectively, for every exact solution $U(t)$ without reference to any particular evolution. However, there exist cases when $U_0(t)$ and $U_1(t)$ are directly related to certain types of evolution: for example, systems governed by dipole Hamiltonians, for which the evolution operator can be decomposed into a combination of unitary operators [22, 23]. These operators, under particular conditions, may be considered as pertaining to either free or interaction evolution. As a criterion for such a classification in the case of weak interactions one might take the characteristic frequency of each evolution mechanism. It would be reasonable to assign fast motion of a system to the free evolution operator $U_0(t)$ and consider slower motion (perturbation) as interaction $U_1(t)$ with an external field.

The meaning of decompositions (13) and (14) can be clearly explained by the example of a spin precession in a magnetic field. Let $U_0(t)$ and $U_1(t)$ describe spin rotations about two instant axes with different angular frequencies: $U_0(t)$ pertaining to fast motion (considered as free evolution) and $U_1(t)$ belonging to slower evolution (interaction). Then evolution operator (13) represents spin precession about the slowly rotating magnetic field direction, the situation which usually occurs in geometric phase experiments with polarized neutrons. More generally, this is an adiabatic approximation. The second case (14) represents nutation of the precession axis in the rotating frame, as in magnetic resonance experiments with a spin subjected to the bias constant and orthogonal harmonic magnetic fields. This is an example of evolution operator decomposition into parts, $U_0(t)$ and $U_1(t)$, each of them representing actual evolution.

The reference basis $\{|m(t)\rangle\}$ is introduced by subjecting the eigenbasis of the free Hamiltonian to the interaction evolution operator $U_1(t)$:

$$|m(t)\rangle = U_1(t)|m\rangle \qquad |m(0)\rangle = |m\rangle. \tag{15}$$

We introduce geometric phase in the noninteraction picture as a difference between the total and dynamical phases, defined with respect to $|m(t)\rangle$:

$$\gamma_m^{(0)}(t) = \arg\langle m(t)|\Psi(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \langle\Psi(t')|H(t')|\Psi(t')\rangle. \tag{16}$$

By specifying a particular basis $\{|m(t)\rangle\}$ we remove the ambiguity in geometric phase definition, which was discussed in section 2. Therefore (16) represents one of the possible choices of geometric phase decompositions in the Hilbert space. Note that in [1], Berry chose $|m(t)\rangle$ to be the cyclic instantaneous eigenstate of the Hamiltonian. If this condition is assumed, expression (16) exactly reduces to that of Berry.

Next we will show that decomposition (16) is equivalent to the general decomposition of geometric phase [21], if it is applied to the state vector in the noninteraction picture. The noninteraction picture is introduced in the opposite way to the interaction picture (see also [3]):

$$|\Psi^{(0)}(t)\rangle = U_1^+(t)|\Psi(t)\rangle \qquad |\Psi^{(0)}(0)\rangle = |\Psi(0)\rangle = |m\rangle \tag{17}$$

$$H^{(0)}(t) = U_1^+(t)H(t)U_1(t). \tag{18}$$

Adopting the general procedure [21], one gets geometric phase decomposition in the noninteraction picture:

$$\gamma_m^{(0)}(t) = \arg\langle\Psi^{(0)}(0)|\Psi^{(0)}(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \langle\Psi^{(0)}(t')|H^{(0)}(t')|\Psi^{(0)}(t')\rangle \tag{19a}$$

$$= \arg\langle m|U_1^+(t)|\Psi(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \langle\Psi(t')|U_1(t')\, U_1^+(t')H(t')U_1(t')\, U_1^+(t')|\Psi(t')\rangle \tag{19b}$$

$$= \arg\langle m(t)|\Psi(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \langle\Psi(t')|H(t')|\Psi(t')\rangle \tag{19c}$$

which coincides with (16). This means that the noninteraction picture allows determination of geometric phase via the common procedure (19*a*) which preserves the form (19*c*) of the free geometric phase (8*a*) and, furthermore, reduces to it when interaction vanishes out. The difference with noninteraction picture is that all characteristics are defined with respect to the time-dependent reference basis $\{|m(t)\rangle\}$. First of all, it modifies the total phase, the first term in (19*c*). The second term in (19*c*), the dynamical phase (with the opposite sign only), is the mean value of the Hamiltonian and therefore is independent of the particular representation. However, dynamical phase depends on the choice of basis $\{|m(t)\rangle\}$, since it specifies the initial state for $|\Psi(t)\rangle$, namely $|m(0)\rangle = |m\rangle$, which in our case is the eigenvector of the free Hamiltonian. Therefore the noninteraction picture affects not only the total phase part in geometric phase decomposition, but also dynamical phase as well.

Now let us study geometric phase decomposition (16) for a particular form of Hamiltonian, corresponding to evolution operators of the form (13) and (14). Evolution operator (13) corresponds to the Hamiltonian

$$H(t) = U_1(t)H_0U_1^+(t) + H_1(t) \tag{20}$$

where $H_1(t) = \mathrm{i}\hbar\, \dot{U}_1(t)U_1^+(t)$. Substituting (20) into (16), one gets

$$\gamma_m^{(0)}(t) = \arg\langle m|U_1^+(t)U_1(t)U_0(t)|m\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t' \langle m(t')|U_1(t')H_0U_1^+(t') + H_1(t')|m(t')\rangle$$

$$= -\frac{E_m t}{\hbar} + \frac{t}{\hbar}\langle m|H_0|m\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t'\langle m|H_1(t')|m\rangle$$
$$= \mathrm{i}\int_0^t \mathrm{d}t'\langle m(t')|\frac{\mathrm{d}}{\mathrm{d}t'}|m(t')\rangle. \qquad (21)$$

This result is similar to that of Berry, released from certain conditions, as was mentioned above. This also coincides with the Floquet decomposition of the geometric phase for cyclic states, if $U_1(t)$ is assumed to be periodic. While being of the same form as the Floquet decomposition, decomposition (16) extends the method to incomplete cyclic evolutions. In this way, it enables examination of the temporal evolution of cyclic geometric phase, the final point of which, $\gamma_m^{(0)}(T)$, coincides with the Floquet result.

The second evolution operator (14) corresponds to the Hamiltonian

$$H(t) = H_0 + U_0(t)H_1(t)U_0^+(t) \qquad (22)$$

for which geometric phase decomposition (16) results in

$$\gamma_m^{(0)}(t) = \arg\langle m(t)|U_0(t)|m(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t'\langle m(t')|H_0 + H_1(t')|m(t')\rangle. \qquad (23)$$

In particular, an important consequence follows from (23) for interactions whose Hamiltonians $H_1(t)$ have no diagonal matrix elements in the eigenbasis $\{|m\rangle\}$ of the free Hamiltonian:

$$\langle m|H_1(t)|m\rangle = 0 \qquad (24)$$

which is equivalent to the parallel transport condition (2). In this case the second term in (23) integral vanishes and geometric phase becomes

$$\gamma_m^{(0)}(t) = \arg\langle m(t)|U_0(t)|m(t)\rangle + \frac{1}{\hbar}\int_0^t \mathrm{d}t'\langle m(t')|H_0|m(t')\rangle. \qquad (25)$$

This is analogous to the geometric phase expression in the absence of interaction (8*a*), except that in (8*a*) vector $|\Psi_0(0)\rangle$, indicating initial state, is time independent, while in (25), instead, the reference basis $|m(t)\rangle$ is time dependent. This means that geometric phase in the noninteraction picture possesses all the characteristics of the free geometric phase. Interaction only modifies the reference basis, which is equivalent to the change of initial state in (8*a*). Since in most cases $|H_1(t)| \ll |H_0|$, $U_1(t)$ represents slower evolution than $U_0(t)$. Therefore the behaviour of geometric phase (25) may be regarded as if determined by free evolution $U_0(t)$ in the slowly varying instantaneous eigenbasis $\{|m(t)\rangle\}$. The condition (24) is frequently encountered. In particular, it is satisfied for a spin in a rotating magnetic field, when resonance is reached, as will be discussed in the following example.

4. Geometric phase for a spin-$\frac{1}{2}$ in a rotating magnetic field

To illustrate some geometric phase properties in the noninteraction picture we will study an example of a spin-$\frac{1}{2}$ particle in a rotating magnetic field, which is a subject of investigations in magnetic resonance theory and which has attracted much attention since the discovery of geometric phase.

Consider a spin-$\frac{1}{2}$ subjected to an external magnetic field $\boldsymbol{B} = (B_1\cos\omega t, B_1\sin\omega t, B_0)$. The corresponding Hamiltonian is given by [24]

$$H(t) = \frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\omega_1}{2}(\sigma_x\cos\omega t + \sigma_y\sin\omega t) \qquad (26)$$

where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices, $\omega_0 = gB_0$, $\omega_1 = gB_1$ and g is the gyromagnetic ratio. The solution of the Schrödinger equation

$$\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \frac{\mathrm{i}}{\hbar}H(t)|\Psi(t)\rangle \qquad (27)$$

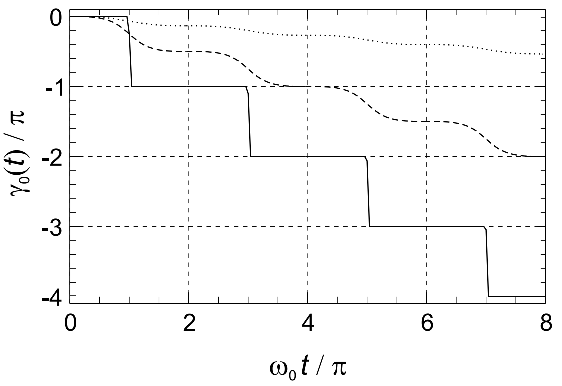

Figure 1. Temporal characteristics of the geometric phase for a spin-$\frac{1}{2}$ in a constant magnetic field for a set of parameters θ of the initial state: 30° (dotted curve), 60° (dashed curve), 90° (solid curve).

is well known [24]:

$$|\Psi(t)\rangle = U(t)|\Psi(0)\rangle = \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}\sigma_z}\mathrm{e}^{-\mathrm{i}\frac{\Omega t}{2}(\boldsymbol{n},\boldsymbol{\sigma})}|\Psi(0)\rangle \tag{28}$$

where $\Omega = (\omega_1^2 + \delta\omega^2)^{1/2}$, $\boldsymbol{n} = \Omega^{-1}(\omega_1, 0, \delta\omega)$, $\delta\omega = \omega_0 - \omega$. The first term in (26) may be considered as a free Hamiltonian $H_0 = \frac{\hbar\omega_0}{2}\sigma_z$, and the second one as that of the interaction. Free evolution is defined by the evolution operator $U_0(t) = \exp(-\mathrm{i}\frac{\omega_0 t}{2}\sigma_z)$, and the free geometric phase for the initial state $|\Psi(0)\rangle = (\cos\theta/2, \sin\theta/2)^T$, according to (8*b*), is

$$\gamma_0(t) = -\arctan\left[\cos\theta\ \tan\frac{\omega_0 t}{2}\right] + \cos\theta\frac{\omega_0 t}{2}. \tag{29}$$

This is an expression of geometric phase for a spin-$\frac{1}{2}$ in a constant magnetic field. The most characteristic features of geometric phase, nonlinearity [25] and phase jumps [26], are illustrated in figure 1. There, geometric phase is plotted over several periods $T_0 = 2\pi/\omega_0$ for different initial conditions in order to compare the characteristics with the case of geometric phase under interaction, which extends over a much longer period $T = 2\pi/\Omega$.

We define the interaction evolution operator in accordance with decomposition (14) as

$$U_1(t) = U_0^+(t)U(t) = \mathrm{e}^{\mathrm{i}\frac{\delta\omega t}{2}\sigma_z}\mathrm{e}^{-\mathrm{i}\frac{\Omega t}{2}(\boldsymbol{n},\boldsymbol{\sigma})}. \tag{30}$$

The choice of decomposition (14) is reasonable in the case of a weak harmonic field $B_1 \ll B_0$ in the vicinity of resonance $|\delta\omega| \ll \omega_0$. In this case, $U_1(t)$ represents rotations with frequencies $\delta\omega$ and Ω, much smaller than ω_0, the frequency of the free evolution. Such a situation normally occurs in magnetic resonance experiments.

Note that an alternative choice of operator decomposition (13)

$$U_1(t) = U(t)U_0^+(t) = \mathrm{e}^{-\mathrm{i}\frac{\omega t}{2}\sigma_z}\mathrm{e}^{-\mathrm{i}\frac{\Omega t}{2}(\boldsymbol{n},\boldsymbol{\sigma})}\mathrm{e}^{\mathrm{i}\frac{\omega_0 t}{2}\sigma_z} \tag{31}$$

would lead to both free and interaction evolution operators having high frequency ω, ω_0 components.

Geometric phase decomposition (16) for the initial state $m = +\frac{1}{2}$ yields

$$\gamma_{1/2}^{(0)}(t) = \beta_{1/2}^{(0)}(t) - \alpha_{1/2}^{(0)}(t) \tag{32}$$

$$\beta_{1/2}^{(0)}(t) = \arg\langle\tfrac{1}{2}|U_1^+(t)U(t)|\tfrac{1}{2}\rangle = \arg\langle\tfrac{1}{2}|\mathrm{e}^{\mathrm{i}\frac{\Omega t}{2}(\boldsymbol{n},\boldsymbol{\sigma})}\mathrm{e}^{-\mathrm{i}\frac{\omega_0 t}{2}\sigma_z}\mathrm{e}^{-\mathrm{i}\frac{\Omega t}{2}(\boldsymbol{n},\boldsymbol{\sigma})}|\tfrac{1}{2}\rangle$$

$$= -\arctan\left[\left(\cos^2\frac{\Omega t}{2} + (n_z^2 - n_x^2)\sin^2\frac{\Omega t}{2}\right)\tan\frac{\omega_0 t}{2}\right] \tag{33}$$

$$\alpha_{1/2}^{(0)}(t) = -\frac{1}{\hbar}\int_0^t \mathrm{d}t' \left\langle\frac{1}{2}\right| U_1^+(t')H(t')U_1(t') \left|\frac{1}{2}\right\rangle$$
$$= -\frac{1}{2}\left[\omega_0\left(n_z^2 t + n_x^2\frac{\sin\Omega t}{\Omega}\right) + \omega_1 n_x n_z\left(t - \frac{\sin\Omega t}{\Omega}\right)\right]. \tag{34}$$

In the case of exact resonance when a perpendicular magnetic field rotates with angular frequency ω equal to a Larmor frequency ω_0, $\delta\omega = 0$, equations (32)–(34) become

$$\gamma_{\mathrm{res}}^{(0)}(t) = \beta_{\mathrm{res}}^{(0)}(t) - \alpha_{\mathrm{res}}^{(0)}(t) \tag{35}$$

$$\beta_{\mathrm{res}}^{(0)}(t) = -\arctan\left[\cos\frac{\omega_1 t}{2}\ \tan\frac{\omega_0 t}{2}\right] \tag{36}$$

$$\alpha_{\mathrm{res}}^{(0)}(t) = -\frac{\omega_0}{2}\frac{\sin\omega_1 t}{\omega_1}. \tag{37}$$

As the interaction vanishes $\omega_1 \to 0$, geometric phase approaches zero and in the limit only the dynamical phase remains: $\beta_{\mathrm{res}}^{(0)}(t) = \alpha_{\mathrm{res}}^{(0)}(t) = \omega_0 t/2$. This situation corresponds to a system evolving in one of its stationary eigenstates, when only dynamical phase may exist.

For the resonant case one may check the validity of the parallel transport condition (24). The interaction evolution operator (30) for $\delta\omega = 0$ turns into $U_1(t) = \exp(-\mathrm{i}\frac{\omega_1 t}{2}\sigma_x)$. The corresponding Hamiltonian $H_1(t) = \frac{\hbar\omega_1 t}{2}\sigma_x$ satisfies (24) because of the orthogonality of Pauli matrices.

From the great variety of evolutions, we make closer examination of the cyclic states, for which an initial state after fixed time T returns to itself up to a phase factor. This imposes restrictions on the evolution operator $U(T)$ in (28):

$$\Omega T = 2\pi p \qquad \text{and} \qquad \omega T = 2\pi q \tag{38}$$

with p and q being integer numbers. Geometric phase dependences on time for cyclic evolutions are shown in figures 2 and 3. There, the time variable and other parameters are normalized to the fundamental period $T = 2\pi/\Omega$ in order to define the parameters in terms of phases. In figure 2 curves are plotted for a set of deviations $\delta\omega$ from the resonant frequency. In principle, what makes the curves different are the Poincaré sphere coordinates $\omega_0 t$ and θ in (29), or the relevant effective parameters in (32)–(34) [26]. While $\omega_0 t$ designates a time moment of geometric phase jump, θ determines the amplitude and slope of the jump. The effect of summation of separate jumps finally determines the shape of the curves. The most characteristic features of geometric phase temporal evolution, namely nonlinearity [25] and phase jumps [26] are also noticeable in the noninteraction picture. The time dependence of the dynamical phase is smoother than that of the geometric phase. As is evident from (34), only linear and sine functions appear in the expression of dynamical phase. Geometric phase for resonant evolutions with $\delta\omega = 0$ is shown in figure 3. One can see a change of geometric phase by $\pm\pi$ after each period. Geometric phase $\gamma_{\mathrm{res}}^{(0)}(t)$ is time periodic $\gamma_{\mathrm{res}}^{(0)}(T) = 0$ (without ambiguity $2\pi k$) only for evolutions with $\omega T = \omega_0 T$ being integer multiples of 4π. This reflects the fact that for such evolutions spin rotates 'up' and 'down' along symmetrical paths with respect to the Poincaré sphere coordinates $\omega_0 t$ and θ. Geometric phase lost by spin rotating 'down' is completely recovered when travelling 'up'. The parallel with 'geometric phase transition' is strengthened by the fact that geometric phase reaches a maximum for the completely nonstationary spin state, i.e. when the probability for the spin to be in one of its two eigenstates is 0.5. That is, spin possesses the greatest ability to perform a transition when

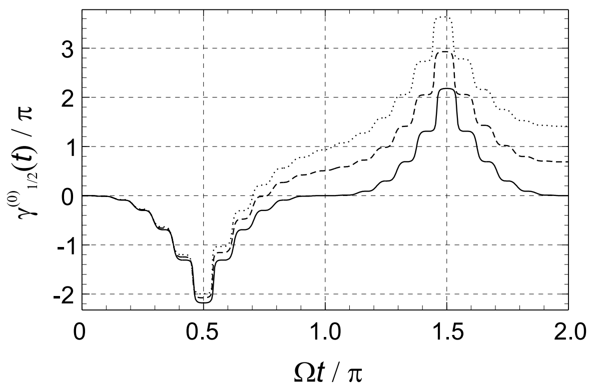

Figure 2. Temporal characteristics of the geometric phase for a spin-$\frac{1}{2}$ in a rotating magnetic field for a set of deviations from resonance $\delta\omega/\omega_0$: 0 (solid curve), 0.007 (dashed curve), 0.01 (dotted curve). Other parameters are $\Omega T = 2\pi$, $\omega T = 48\pi$.

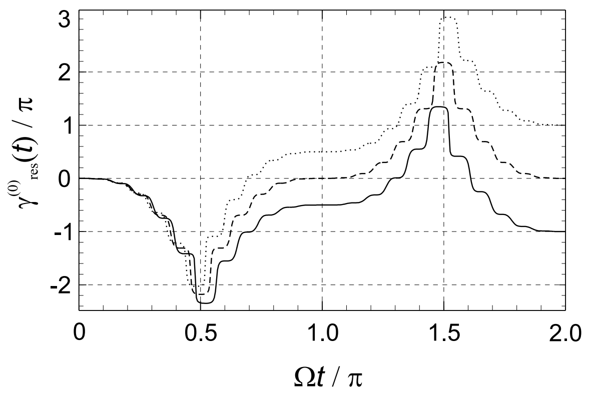

Figure 3. Temporal characteristics of the geometric phase for a spin-$\frac{1}{2}$ in a rotating magnetic field for the resonant case $\delta\omega = 0$, $\Omega T = \omega_1 T = 2\pi$ and a set of resonant frequencies $\omega T = \omega_0 T$: 46π (solid curve), 48π (dashed curve), 50π (dotted curve).

geometric phase is maximal. The probability for the spin being initially in the $m = +\frac{1}{2}$ state to complete a transition to the $m = -\frac{1}{2}$ state at the time moment t is

$$P_{-\frac{1}{2},+\frac{1}{2}}(t) = |\langle -\tfrac{1}{2}|U(t)| + \tfrac{1}{2}\rangle|^2 = n_x^2 \frac{1 - \cos\Omega t}{2} \tag{39}$$

which in the case of resonance ($n_x = 1$) equals 0.5 for time moments $\Omega t = \{\pi/2, 3\pi/2\}$. As is evident from figure 3, these are the maximum points of the geometric phase for the resonant case $\omega T = \omega_0 T = 48\pi$. The geometric phase connection with the nonstationarity of a quantum state for complete paths is already known. It was shown in [27] that geometric phase is maximal for cyclic evolutions, for which initial and final states are completely nonstationary. The noninteraction picture extends the relationship between geometric phase and nonstationarity of state to incomplete cyclic, as well as to general noncyclic evolutions. It therefore enables investigations of geometric phase temporal characteristics.

5. Conclusions

The proposed decomposition of geometric phase is based on the separation of the system's evolution operator into parts pertaining to the free motion and interaction. While eliminating ambiguity in geometric phase decomposition, such a partition also manifests the arbitrariness due to the freedom in selecting which part of evolution should be assigned to free motion and which to interaction. The method does not impose any restrictions on the form of these separate operators of evolution, except that they must be known prior to the evaluation of geometric phase. It means that the free evolution, as well as being perturbed, must represent exactly solvable problems. In the case of weak interactions a choice of (13) or (14) type of separation of evolution operator can be made based on the characteristic frequency of the particular evolution operator. The procedure can be extended for a number of interactions following each other in the order (13) or (14). In this case, geometric phase can be defined with respect to each successively applied interaction. The method provides a procedure to compare geometric phases for the successive decompositions (cf (8*a*) and (25)). It therefore enables investigation of the influence of a particular interaction on the geometric phase. As is shown in the example, for particular cases, geometric phase is closely related to the transition induced by the corresponding interaction.

Note added in proof. After the paper was finished, we were pointed to the problem of different dimension spaces associated with two types of geometric phase: (i) adiabatic Berry phase [1] resulting from a closed path in some parameter space and (ii) cyclic Aharonov–Anandan phase [2] resulting from a closed path in the parametric Hilbert space. This also falls within the scope of applications of our method since it can be used in both adiabatic and nonadiabatic cases, as illustrated by the given above example of a spin-$\frac{1}{2}$ particle in a magnetic field. Indeed, for a particle of any spin in a rotating magnetic field, adiabatic Berry phase is defined in the 2-sphere magnetic field space. While for a particle of spin s in a constant magnetic field, Aharonov–Anandan phase is associated with $2s$-dimensional projective Hilbert space [28], which is a 2-sphere only for $s = \frac{1}{2}$ particles. It is well known [2, 6, 8] that geometric phase is due to the holonomy of a connection in a principal fibre bundle over the projective Hilbert space of rays, and depends only on the path traced by the ray and the curvature of the ray space. It may seem that geometric phase cannot be defined for spin in rotating and constant magnetic fields simultaneously in the same framework, since it involves different projective spaces. However, we suppose that this is a matter of different approaches used to define geometric phase. As has been noted earlier [17], geometric phase can be calculated in two ways: (i) by a pure geometric method describing a ray path in the projective Hilbert space, or alternatively, (ii) by determining the dynamics, which generates the ray. We have chosen the second alternative, the dynamical approach, based on the evolution operator method, previously developed and used by a number of authors [4, 5, 7, 10]. This approach does not give insight into the topological properties of geometric phase spaces, but it allows us to treat different kinds of dynamics in the unified way. To merge both geometric and dynamical approaches, a more general theory is required, perhaps one that includes group-theoretical description similar to [8].

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